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Some Remarks on Prime Submodules in Weak Co-Multiplication Modules

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ABSTRACT
Throughout this paper, the entire ring will be treated as commutative ring with non-zero identity and all the modules will be treated as unitary modules. In this paper we shall discuss some remarks on prime submodules in weak co-multiplication modules.
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I. INTRODUCTION

In the theory of modules, prime submodules of modules are very important concept in an algebraic structure and have obtained a very important attention in the society of research of this field.

In 1981, Barnad [26] has given the concept of multiplication modules. After using the concept prime submodules of modules, the concept of weak multiplication module has been given by Azizi Shiraz [6] in the year 2003. Ansari-Toroghy and Farshadifer [8] has introduced the notion co-multiplication module as a dual notion of multiplication module in 2007. Using the concept of prime submodule of modules the concept of weak co-multiplication module has been given by Atani and Atani [7] in the year 2009. So in that way we can say that the notion co-multiplication module is the dual of the notion weak multiplication module. The aim of this paper is to provide more information in this field and to discuss some results on prime submodules in weak co-multiplication modules.

II. PRELIMINARIES

Throughout this paper, we consider all rings as commutative rings with identities and all modules as unital modules. The notation Z will denote the ring of integers and Q will denote the ring of rational numbers. In this section, we give some basic definitions and results for further understanding.

If N and K are submodules of R-module M then the **residual ideal** N by K is defined as an ideal $(N :_R K) = \{r \in R \mid rK \subseteq N\}$.

In a special case in which N= 0, the ideal $(0 :_R K)$ is called annihilator of K and it is denoted by $Ann_R(K)$. Also the submodule $(0 :_M I)$ is called the annihilator of I in M and it is denoted by $Ann_n(I)$.

If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R \mid rM \subseteq N\}$ will be denoted by $[N : M]$. Then $[0 : M]$ is the annihilator of M, Ann (M).

Definition 2.1 [1]: An R-module M is called a **multiplication module** if for each submodule N of M, N = IM, for some ideal I of R. In this case we can take I = [N : M].

It is easy to see that in this case N = [N: M] M, where [N : M] = Ann (M/N). (see [17])
Clearly, M is a multiplication module if and only if for each $m \in M$, $Rm = [Rm : M]M$. (see [2])
For an R-module M, we define the ideal $\theta(M) = \sum_{m\in M}[Rm : M]$.

If M is multiplication then
$$M = \sum_{m\in M} Rm = \sum_{m\in M}[Rm : M]M = \left(\sum_{m\in M}[Rm : M]\right)M = \theta(M)M.$$

Moreover, if N is a submodule of M, then
$$N = [N : M]M = [N : M]\,\theta(M)M = \theta(M)[N : M]M = \theta(M)N. \quad \text{(See [3])}$$

Example 2.2 [4]: Examples of multiplication ideals (i.e. ideals of a rings R that are multiplication R-modules) include invertible ideals, principal ideals and ideals generated by idempotent.

Sang Cheol Lee, Sunah Kim and Sang-Cho Chung [5] has defined multiplication module in the term of extended as follows:

Definition 2.3 [5]: Let R be a commutative ring and let M be an R-module. Then a submodule N of M is said to be **extended** if N = IM, for same ideal I of R.

Definition 2.4 [5]: M is called multiplication module if every submodule of M is extended. For example, every proper submodule of the Z-module $Z(p^\infty)$ is a multiplication module but the Z-module $Z(p^\infty)$ itself is not a multiplication module.

In H. Ansari - Toroghy and F. Farshadifar [8] the notion of a **co-multiplication module** was introduced as a dual of the concepts of a multiplication module.

Definition 2.5 [5]: An R-module m is defined to be a **co-multiplication module** if for each submodule N of M, N = $(0 :_M I)$ for same ideal I of R.
In this care we can take I = Ann (N).

Example 2.6: The Z-module Z_{2^∞} is co-multiplication module since all of its proper submodules are of the form $(0 :_M 2^k Z)$ for K = 0,1,2,3,4,……
It is clear that M is co multiplication module if and only if for every submodule N of M, we have $Ann_M (Ann_R (N)) = N$. (see[9])

Example 2.7: Z_4 is a co-multiplication Z-module. Consider the Z-module M = Z / 4Z and set N = 2Z / 4Z. Then N and M/N are co-multiplication Z-modules. (see [20])

An R-module M is called co-multiplication module if for any submodule N of M, there exists an ideal I of such that N = $Ann_M(I)$. (see [11] [12])
An R-module M is co-multiplication module if and only if for any submodule N of M, N = $(0 :_M Ann (N))$ (see [8])
Every proper submodule of a co-multiplication module is co-multiplication module. (see [8])

Example 2.8: If V is a two dimensional vector space over a filed K then V cannot be co-multiplication module but every proper submodule as everyone-dimensional vector space is co-multiplication module. (see [11])

Example 2.9 [8]: Let p be any prime number. Let M = $Z(p^\infty)$. Then M is a co-multiplication Z-module where Z is the ring of integers.

Proof: Fix a prime integer p. Define a set
$Q_p = \{ r / p^t \mid r, t \in Z\}$
Then, Q_p is additive abelian group containing Z. Define a set M = $Z(p^\infty)$ = Qp / Z.
Then, M is a Z- module. Let N be any submodule of M. Then N = $Z(1/p^i + Z)$, for some integer i.($i \geq 0$).
Set I = $p^i Z$.
Now, $Ann_M (p^i Z)$ = N. Hence M is a co-multiplication Z-module.

Now here is an example which shows that not every R-module is co-multiplication module.

Example 2.10 [11]: Consider Z as a Z-module. Now, 2Z is a submodule of Z, we have
$Ann_Z(2Z) = \{m \in Z \mid 2Zm = 0\} = (0)$.
Now, if Z is a co-multiplication module then by [8], we have
$2Z = (0 :_Z Ann_Z (2Z))$
But $(0 :_Z Ann_Z (2Z)) = (0 :_Z 0) = Z \neq 2Z$
Therefore, Z is not a co-multiplication module.

As in [8] the notion of co-multiplication module was introduced as a dual of the concept of a multiplication module. An R-module M is called co-multiplication (co-m for short) if for every submodule N of M, there exists an Ideal I of R such that N = $(0 :_M I)$.

It is clear that M is co-m if and only if for every submodule N of M, we have $Ann_M (Ann_R (N)) = N$.

Definition 2.11 [10]: A proper submodule N of M is said to be **prime submodule** if from r m $\in$ N for r $\in$ R and m $\in$ M, We can deduce that r $\in$ (N : M) or m $\in$ N. (see for example in [10])

Definition 2.12 [6]: An R-module M is called **weak multiplication module** if M doesn't have any prime submodule or every prime submodule N of M We have N = IM where I is an ideal of R.

Example 2.13 [6]: Q is a weak multiplication Z-module, which is not a multiplication Z-module.
Every finitely generated weak multiplication module is a multiplication module. (see [6]).

In 2009 Reza Ebrahimi Atani and Shahabaddin Ebrahimi Atani [7] have given the notion **weak co-multiplication modules** as the dual notion of weak multiplication modules and studied on weak co-multiplication modules.

Definition 2.14 [7]: Let R be a commutative ring. An R-module M is defined to be a **weak co-multiplication module** if M doesn't have any prime submodule or for every prime sub module N of M, N = $(0 :_M I)$ for same ideal I of R.
One can easily show that if M is a weak co-multiplication module, then N = $(0 :_M Ann(N))$ for every prime submodule N of M.
We note that every co-multiplication module is weak co-multiplication module. (see [19])
It should be mentioned that another dual notion of weak multiplication module is defined in [19] as second submodule.

Definition 2.15 [19]: A submodule N of M is said to be a **second submodule** if $rN = N$ or $rN = 0$ for each $r \in R$.

Definition 2.16 [19]: An R-module M is a **weak co-multiplication module** if M does not have any **second submodule** or for every second submodule or for every second submodule S of M, we have $S = (0 :_M I)$, where I is an ideal of R.

Remark in H. Ansari-Toroghy and F. Farshadifar [19] it is clear that every co-multiplication R-module is a weak co-multiplication R-module. However the converse is not true in general.
For example, the Z-module Q (Here Q denotes the field of rational numbers) is a weak co-multiplication Z-module while it is not a co-multiplication Z-module.

In H. Ansari-Toroghy and F. Farshadifar [19] M is called weak co-multiplication module when for each **second submodule** N of M, there is an ideal I of R, such that $N = (0 :_M I)$ but here we are the term weak co-multiplication module exclusively in sense of [7] only.

Definition 2.17 [13]: Let R be a commutative ring. An R-module L is said to be **co-cyclic** if $L \subseteq E(R|P)$ for some maximal ideal P of R. (See [13] and [14])

Proposition 2.18 [8]: Every co-cyclic module over a commutative complete Noetherian ring is a co-multiplication module.
Proof: (See [8])

Definition 2.19 [15]: An R-module M is said to be semi simple module (resp. co-semisimple module) in case every submodule of M is the sum (resp. intersection) of minimal (resp. maximal) submodule. (Also See [16])

III. MAIN RESULTS

There is a large body of researches concerning with multiplication modules. It is natural to ask the following question: to what extent does the dual of their results hold for co-multiplication modules. Similar questions arise for weak multiplication modules and weak co-multiplication modules. The purpose of this paper is to obtain more information about this class of modules.

Proposition 3.1 [8]: every co-cyclic module over a commutative complete Noetherian ring is a co-multiplication module.

Remark 3.2: Since every co-multiplication module is weak co-multiplication module therefore every co-cyclic module over a commutative complete Noetherian ring is a weak co-multiplication module.

Proposition 3.3 [15]: Let M be an R-module and if M is co-semi simple module such that $\text{Ann}_R(N) \neq \text{Ann}_R(M)$ for every maximal submodule N of M, then M is co-multiplication module.

Remark 3.4: Since every co-multiplication module is weak co-multiplication module. Therefore, let M be an R-module and if M is co-semisimple module such that $\text{Ann}(N) \neq \text{Ann}(M)$ for every maximal submodule N of M then M is weak co-multiplication module.

Proposition 3.5 [15]: Let M be an R-module if R is Noetherian ring then every injective multiplication R-module is co-multiplication module.

Remark 3.6: Since every co-multiplication module is weak co-multiplication module. Therefore, Let M be an R-module if R is Noetherian ring then every injective multiplication R-module is weak co-multiplication module.

As multiplication modules have been studied by many mathematicians from different point of view. A most suitable reference is Z. A. EI-Bast and P. F. Smith [2]. Particularly in H. Ansari - Toroghy and F. Farshadifar [8] have introduced co-multiplication modules as a dual to multiplication modules. Under various conditions co-multiplication modules are cyclic. (See [18])

Lemma 3.7 [18]:
(1) If a submodule N of M equals $(0 :_M I)$ for same ideal I of R then $(N : M) = (\text{Ann}(M) :_R I)$
(2) M is a weak co-multiplication R-module if and only if it is a weak co-multiplication (R/Ann (M)) - module.
(3) Suppose that M is an R-module and $R = R_1 \times R_2$, where R_1 and R_2 are non- trivial rings. Then $M = M_1 \oplus M_2$ where M_1 is an R_1 module and M_2 is an R_2-module. Also in this case M is weak co-multiplication module if and only if both M_1 and M_2 are weak co-multiplication modules.

Proof: (see [18])

Thus the results for weak co-multiplication modules follow the similar proof of co-multiplication modules.

According to above lemma 3.7, when M is a co-multiplication module, it may be reduced to a faithful co-multiplication module. (See [18])

Similarly when M is a weak co-multiplication module, it may be reduced to a faithful weak co-multiplication module.

Theorem 3.8 [18]: If M is a faithful weak co-multiplication R-module with a maximal submodule

N and R is a reduced ring with decomposition as a finite direct product of indecomposable ring, then M $\cong$ R and R is semi-simple.
Proof: See [18]

Corollary 3.9 [18]: Assume that M is a weak co-multiplication module having a maximal submodule (for example, if M is finitely generated) and Y = Ann (M). Then Y is a prime ideal if and only if Y is a maximal ideal and M $\cong$ R / Y is a simple module.

Corollary 3.10 [18]: If M is a finitely generated co-multiplication module with Ann (M) a radical ideal, then M is cyclic and R / Ann (M) is a semisimple ring.
Proof: see [18].

Definition 3.11 [18]: A chained ring is a ring in which every two ideals are comparable. For example, localization of Z at any prime ideal or more generally every valuation domain is a chained ring.

Lemma 3.12 [18]: If R is a chained ring and M is a co-multiplication module having a maximal submodule N, then M is **cyclic.**
Proof: see [18].

Remark 3.13: Since every co-multiplication module is weak co-multiplication module. (see [19]), so for the above results, under the various conditions, weak co-multiplication modules are also cyclic.

Thus since every co-multiplication module is weak co-multiplication module, so for the above results, under the various conditions, weak co-multiplication modules are also cyclic.

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